Tao, Section 1.3

V is a Hilbert space:

1) Vector space over complex #s

   2 operations $+: V \times V \to V$, $\cdot : \mathbb{C} \times V \to V$

2) Positive definite product
3) Complete wrt metric induced by inner product.

Usual conditions. Addition of an inner product

- $(y,x) = \overline{(x,y)}$
- Linear $(ax_1 + bx_2, y) = a(x_1, y) + b(x_2, y)$
- Positive definite: $(x,x) \geq 0$ with equality iff $x = 0$

☆ What happens to

$$(y, ax + by) = ?$$

(its antilinear)

Theorem 1.3 (Spectral Theorem): V finite dim Hilbert space, $T : V \longrightarrow V$ self adjoint operator.

Then, $\exists\ \lambda_1, \ldots, \lambda_N$ and $v_1 \ldots v_N$

st $$T v_i = \lambda_i v_i$$ and $v_1 \ldots v_N$ form an

ON basis.

Pf: By induction. $n=1$ obvious.

$$\underset{|u|_2 = 1}{\operatorname{argmax}}\ Re(u^* T u) = v \qquad \text{(by compactness exists)}$$

v is a critical point of $Re(u^* T u) - \lambda(u^* u - 1)$

by method of Lagrange mult. $\Rightarrow$

$$\frac{d}{dt} Re((v + tg)^* T (v + tg)) - \lambda\left((v+tg)^* (v+tg) - 1\right)\Big|_{t=0} = 0$$

$$\Rightarrow g^* T v + v^* T g - 2\lambda g^* v = 0$$

$$v^* T g = \overline{g T v^*} = g^* T^* v = g^* T v$$

$$g^*(Tv - \lambda v) = 0$$

If this is to be true for all g, Then certainly implies eval equation.

$v^\perp = \{w : w^* v = 0\}$ is preserved by T.

Ex: $v^* T w = w^* T v = 0 \Rightarrow Tw \in v^\perp$

But you can restrict T to $v^\perp$ and this is a lower dim space. Done!

Rem: Can always diagonalize H st H is diagonal, by making a change of basis.

Ex: Show change of basis can be represented as

$\Lambda = V^* H V$ (similarity transformation.)

Q: Are eigenvectors unique?

A: Upto multiplication by a $e^{i\theta}$ a "complex phase"

Q: Is $A \hookrightarrow \lambda_i(A)$ linear?

Is $A \hookrightarrow \lambda_i(A)$ convex? (λ_1 is)

Is $A \hookrightarrow \lambda_i(A)$ concave? (λ_N is)

Theorem: (Courant - Fisher minimax formula)

$$\lambda_i(A) = \sup_{\dim(V)=i} \inf_{\substack{|v|=1 \\ v\in V}} v^* A v \qquad \text{☆3a}$$

$$\lambda_i(A) = \inf_{\dim(V)=n-i+1} \sup_{\substack{|v|=1 \\ v\in V}} v^* A v \qquad \text{☆3b}$$

Pf: By spectral theorem WLOG A is diagonal.
$A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$

Choose $V = \mathrm{span}\{e_1, \ldots e_i\}$

$v = \sum a_i e_i$, $v^* A v = \sum_{k=1}^{i} |a_k|^2 \lambda_k$

$$\Rightarrow \quad \lambda_i(A) \leq \sup_{V} \inf_{|v|=1}$$

Since for this space, obviously the inf is taken at $a_i = 1$.

$$\lambda_i(A) \geq \sup_{V} \inf_{|v|=1} \cdots$$

Let $W = \text{span}\{e_i \cdots e_n\}$

$\dim(W) = n - i + 1$

$U+V$
$\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)$

Claim: Any V of $\dim(V) = i$ must intersect W.

★ Fun exercise, can be done by using $W \oplus W^{\perp} = \mathbb{C}^n$

Given the claim,

$$\inf_{\substack{|v|=1 \\ V}} v^* A v \leq \inf_{\substack{|v|=1 \\ V \cap W}} v^* A v$$

If $v \in W$ then $v = \sum_{k=1}^{n} a_k e_k$

$$v^* A v = \sum |a_k|^2 \lambda_k \leq \lambda_i \sum |a_k|^2 = \lambda_i(A).$$

So for any V of $\dim(V) = i$, $\exists\, v$ st $v^* A v \leq \lambda_i(A)$. Done

Can also use Rayleigh quotient:

$$\lambda_i(A) = \sup_{\dim(V)=i} \inf_{v \neq 0} \frac{v^* A v}{\|v\|^2}$$

More convex & concave quantities

Partial trace: Let A be $n \times n$ Hermitian. Define:

$$tr(A|_V) = \sum_{i=1}^{n} v_i^* A v_i$$

where $\{v_1, \dots v_k\}$ is an ON basis of V

This is basis invariant: If $\{u_1 \dots u_k\}$ another ON basis

Write each $u_i = \sum_{j=1}^{k} c_{ij} v_j = \sum_{j=1}^{k} (u_i, v_j) v_j$

$$\sum_i u_i^* A u_i = \sum_{ij} \overline{c_{ij}} v_j^* A \sum_{ik} c_{ik} v_k = \sum_{ijk} \overline{c_{ij}} v_j^* A c_{ik} v_k \quad \star 5$$

Similarly $v_i = \sum_{j=1}^{k} (v_i, u_j) u_j = \sum_{j=1}^{k} \overline{c_{ji}} u_j$

$$v_j^* v_k = \left(\sum_i \overline{c_{ij}} u_i^*\right)\left(\sum_s c_{sk} u_s\right)$$

$= \sum_i \overline{c_{ij}} c_{ik}$ (using $u_i^* u_s = 0$ if $i \neq s$

$= 0$

$tr(AB) = tr(BA) = \sum_{i,j=1}^{n} A_{ij} B_{ji}$ gives everything.

(*5) $\sum_{i,j,k} \overline{c_{ij}}\, v_j^* A\, c_{ik} v_k = \sum v_j^* A v_j$

☆ This is a good exercise as well.

Prob:

$$\lambda_1(A) + \cdots + \lambda_k(A) = \sup_{\dim(V)=k} tr(A|_V)$$

$$\lambda_{n-k+1}(A) + \cdots + \lambda_n(A) = \inf_{\dim(V)=k} tr(A|_V)$$

☆ : $\lambda_1 + \cdots + \lambda_k$ is a convex fn of A.

Pf: Assume $A = \operatorname{diag}(e_1 \cdots e_n)$

Then choose $V = \operatorname{span}(e_1 \cdots e_k)$

$$tr(A|_V) = \sum_{i=1}^{k} \lambda_i \leq \sup_{\dim(V)=k} tr(A|_V)$$

The rest of the proof is by induction. To show

$$\sum_{i=1}^{k} \lambda_i \geq \sup_{\dim(V)=k} tr(A|_V)$$

$n = 1$ is trivial

Take any V of dim k. Restrict A to $\{e_2, \ldots, e_n\}$

Then it must have V', a $k-1$ dim subspace contained in

$\{e_2, \ldots, e_n\}$. How?

lem: Suppose V and $W \subset \mathbb{C}^n$ are subspaces st

$$\underset{p}{\underset{\parallel}{\dim(V)}} + \underset{q}{\underset{\parallel}{\dim(W)}} = r > n$$

Then $\dim(V \cap W) \geq r - n$

Pf: Only need $V \cap W$ is a subspace. (And the general dimension formula for sums.)

$$\mathbb{C}^n = (V \cap W) \oplus (V \cap W)^\perp \cap V \oplus (V \cap W)^\perp \cap W$$

$$\oplus \quad \left[(V \cap W) \oplus (V \cap W)^\perp \cap V \oplus (V \cap W)^\perp \cap W\right]^\perp$$

$$n = \dim(V \cap W) + p - \dim(V \cap W) + q - \dim(V \cap W) + \dim(\text{rest})$$

$$\Rightarrow \dim(V \cap W) = r - n + \dim(\text{rest}) \geq r - n$$

Then take this V'

$$\lambda_2 + \cdots + \lambda_{k-1} \geq tr(A|_{V'})$$

$$V' \oplus (V'^{\perp_V}) = V$$

← orthogonal complement of V' in V

Or can just take $(V')^{\perp}$ and intersect it with V.
$\dim(V')^{\perp} = n-k+1$
$\dim(V) = k$
$\dim(V + (V')^{\perp}) = n$
$\Rightarrow \dim((V')^{\perp} \cap V) = 1$

Take w in there, unit vector that's a basis.

$$\lambda_1(A) \geq w^* A w \quad \text{by Courant-Fisher}$$

$$\Rightarrow \quad \lambda + \cdots \lambda_{k-1} \geq tr(A|_V) \quad \text{done!}$$

Schur-Horn inequalities: Choose $V = sp\{e_{i_1}, \ldots, e_{i_k}\}$ any k standard basis vectors

$$\lambda_{n-k+1}(A) + \cdots \lambda_n(A)$$

$$\leq a_{i_1 i_1} + \cdots + a_{i_k i_k} \leq \lambda_1(A) + \cdots + \lambda_k(A)$$

★ Ex: Show that these inequalities $\binom{n}{k}$ many are equivalent to showing that the vector

$$\vec{a} = diag(A) = (a_{11}, \ldots, a_{nn}) \text{ lies in the}$$

permutahedron of $(\lambda_1, \ldots, \lambda_n)$, ie the convex hull of the $n!$ permutations of $(\lambda_1, \ldots, \lambda_n)$ in $\mathbb{R}^n$

Ex: Tao says the above fact is related to

Schur-Horn theorem

Atiyah convexity theorem (also Guillemin - Sternberg)

geometric invariant theory in algebraic geometry

I dont know about any of this. You can tell me more during student presentations.

Ex: (doesn't seem hard) (Wielandt minimax formula)

Fix $1 \leq i_1 < \cdots < i_k \leq n$

Partial flag: $V_1 \subset V_2 \cdots \subset V_k$

st $\dim(V_j) = i_j$

Schubert variety: $X(V_1, \dots, V_k)$

$$= \{ W \subset \mathbb{C}^n : \dim(W \cap V_j) \geq j \}$$

$$\lambda_{i_1} + \cdots + \lambda_{i_k} = \sup_{V_1 \cdots V_k} \inf_{W \in X(V_1, \dots V_k)} \operatorname{tr}(A|_W)$$

We generally would like to additively perturb A by B and see how the eigenvalues change.

Use

$$v^*(A+B)v = v^*Av + v^*Bv$$

$$\operatorname{tr}(A+B|_V) = \operatorname{tr}(A|_V) + \operatorname{tr}(B|_V) \quad \text{for any } V \subset \mathbb{C}^n$$

Prop: (Ky-Fan inequality)

$$\sum_{i=1}^{k} \lambda_i(A+B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B)$$

Pf: Fix $\dim(W) = k$

$$\operatorname{tr}((A+B)|_W) = \operatorname{tr}(A|_W) + \operatorname{tr}(B|_W)$$

$$\leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B)$$ Take sup over W.

Fix $1 \leq i_1 < i_2 \cdots < i_k \leq n$

Fix a partial flag $V_1 \ldots V_k$ and $W \in X(V_1 \ldots V_k)$

$$\operatorname{tr}((A+B)|_W) \leq \operatorname{tr}(A|_W) + \sum_{i=1}^{k} \lambda_i(B)$$

Take inf over W, and then sup, and apply Weilandt-Hoffman minimax formula

$$\mathrm{tr}\left((A+B)|_W\right) \leq \mathrm{tr}(A|_W) + \sum_{i=1}^{k} \lambda_i(B)$$

$$\inf_W \left[\mathrm{tr}\left((A+B)|_W\right)\right] \leq \inf_W \left[\mathrm{tr}(A|_W)\right] + \sum_{i=1}^{k} \lambda_i(B)$$

Taking sup over $V_1 \ldots V_k$ gives

Prob (Lidskii inequality):

$$\sum_{j=1}^{k} \lambda_{i_j}(A+B) \leq \sum_{j=1}^{k} \lambda_{i_j}(A) + \sum_{i=1}^{k} \lambda_i(B)$$

Norms: operator & Schatten.

Prop: If A is Hermitian, then individual evals are controlled!

$$|\lambda_i(A+B) - \lambda_i(A)| \leq \|B\|_{op} \leftarrow$$ This is what we want to get to

$$\|A\|_{op} = \sup_{|v|_2=1} |Av| = \sup_{|v|_2=1} \left| \begin{pmatrix} v_1\lambda_1 \\ \\ v_n\lambda_n \end{pmatrix} \right|$$ assuming A is diagonal

$$= \sup_{|v|_2=1} \sqrt{\sum_{i=1}^{n} \lambda_i^2 |v_i|^2}$$ using $|x| = \sqrt{x^* x}$

$$= \max(\lambda_1(A), \lambda_n(A))$$

Ex: If A is Hermitian, show $\|A\|_{op} = \max(\lambda_1(A), \lambda_n(A))$

$$|v^* A v| \leq |v^*| |Av| \leq \|A\|_{op} = \max(|\lambda_1(A)|, |\lambda_n(A)|)$$

$$\lambda_i(A+B) \leq \lambda_i(A) + \lambda_1(B) \qquad \text{(Lidskii)}$$

$$\Rightarrow \quad |\lambda_i(A+B) - \lambda_i(A)| \leq \|B\|_{op}$$

Ex: Show that the operator norm is a bonafide norm on the space of matrices.

Prop: Weyl inequality

$$\lambda_{i+j-1}(A+B) \leq \lambda_i(A) + \lambda_j(B)$$

Pf: Enough to show $\forall$ V st $\dim(V) = i+j-1$

$$\exists\, v, \ |v| = 1 \qquad v A^* v \leq \lambda_i(A) + \lambda_j(B)$$

This follows from the CF minimax formula

$$\lambda_i(A) = \sup_{\dim(V)=i} \inf_{\substack{|v|=1 \\ v\in V}} v^* A v$$

$$\lambda_i(A) = \inf_{\dim(V)=n-i+1} \sup_{\substack{|v|=1 \\ v\in V}} v^* A v$$

Find U of codimension $i-1$ $(\dim(U) = n-i+1)$

(take U to be the eigenspace corresponding to $(\lambda_{n-i+1}, \ldots, \lambda_n)$)

$$\lambda_i(A) \geq \sup_{\substack{|v|=1 \\ v\in U}} v^* A v$$

Similarly $\exists\, W$ $\dim(W) = n-j+1$

$$\lambda_j(A) \geq \sup_{\substack{|v|=1 \\ v\in W}} v^* A v$$

$\dim(U^{\perp} + W^{\perp}) \leq i+j-2$

$\Rightarrow \dim(U^{\perp} \cap W^{\perp})$

$$\dim(U^{\perp}) + \dim(W^{\perp}) = i+j-2$$

$$\Rightarrow \dim((U\cap W)^{\perp}) \leq \dim(U^{\perp} + W^{\perp}) \leq i+j-2$$

$$(U\cap W)^{\perp} \subseteq U^{\perp} + W^{\perp}$$

$$\Rightarrow \quad \dim(U\cap W) \geq n-(i+j-1)+1$$

$$\dim(V) = i+j-1$$

But $\dim(U\cap W) + \dim(V) = n+1 > n$

So $\dim(U\cap W\cap V) \geq 1$

So we have found a vector in V that does the job.

★ 1.3.6 Ex

$$\sum_{i=1}^{n} c_i \lambda_i(A+B) \leq \sum_{i=1}^{n} c_i \lambda_i(A) + \sum_{i=1}^{n} c_i^* \lambda_i(B)$$

for $c_i \geq 0$ and $c_1^* \geq \cdots \geq c_n^* \geq 0$ decreasing rearrangement.

$$\int_0^\infty \mathbb{1}_{\{c_i \geq x\}} \, dx = c_i$$

Then $$\sum_{i=1}^{n} c_i \lambda_i(A+B) = \int_0^\infty \sum_{i=1}^{n} \mathbb{1}_{\{c_i \geq x\}} \lambda_i(A+B) \, dx$$

Let $\vec{i}(x) = \{ i \in [n] : c_i \geq x \}$

$$= \int_0^\infty \sum_{j \in \vec{i}(x)} \lambda_j(A+B) \, dx$$

$$\leq \int_0^\infty \sum_{j \in \vec{i}(x)} \lambda_j(A) \, dx + \sum_{i=1}^{|\vec{i}(x)|} \lambda_i(B) \, dx \quad dx$$

and now see when $|\vec{i}(x)| = 1$, 2 and so on. Now you see where the decreasing rearrangement comes from.

$$\sum_{i=1}^{n} c_i \left(\lambda_i(A+B) - \lambda_i(A)\right) \leq \sum_{i=1}^{n} c_i^{*} \lambda_i(B) \quad — \text{☆5}$$

Taking a sup over c_i st $\|(c_1, \dots, c_n)\|_{\ell q}$ gives $\|\lambda_i(A+B) - \lambda_i(B)\|_{\ell p}$

The same sup applies on the RHS $(\lambda_1 \geq \lambda_2 \dots)$ and $, \geq \dots)$ apply and you get

Ex: $\|v\|_p = \sup_{\|u\|_q = 1} (u, v)$ $q = \frac{p}{p-1}$

$u = (v_j^{p/q} \operatorname{sgn}(v_j))$ is the optimizer

$$(☆5) = \|(\lambda_i(B))_{i=1}^{n}\|_{\ell p} . = \|B\|_{S^p}$$

Called Schatten-p norm.

→ Is a bonafide norm on the space of Hermitian matrices.

→ What is $\|B\|_{S^\infty}$?

→ $\|A\|_{S^2} = \|(\lambda_i(A))_{i=1}^{n}\|_{\ell 2}$ Hilbert-Schmidt norm.

$$tr(AA^*) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \overline{a_{ij}} = \sum_{i,j} |a_{ij}|^2$$

$$(AA^*)_{ii} = a_{ij} A^*_{ji} = a_{ij} \overline{a_{ij}}$$

If A is Hermitian we may as well assume A is diagonal and thus,

$$tr(AA^*) = \|A\|_{S^2}$$

In the $p=2$ case

$$\sum_{i=1}^{n} |\lambda_i(A+B) - \lambda_i(A)|^2 \le \|B\|_{S^2}^2$$

Ex: Let S be the set of real symmetric matrices. dim(S)

$= n + \binom{n}{2}$

Let R = set of matrices with repeated eigenvalue.

Show $R^c \subset V$, $\dim(V) = 2$

How to parametrize S ?

Pick λ_1. Then pick v_1 st $\|v\|_1 = 1$ (n parameter)

Then pick λ_2, and v_2 st $v_1 \cdot v_2 = 0$ and $\|v_2\| = 1$

This is another $n-1$ parameters and so on.

$\lambda_1 \to n$
$\lambda_2 \to n-1$
$\vdots$
$\lambda_n \to v_n$ completely determined. 1

Total $= n + n-1 + \cdots + 1 = \frac{n(n+1)}{2}$

Now suppose eigenvalues are repeated. How do you deal with it?

wlog, we can assume its the last two that are repeated.

$\lambda_1 \cdots\cdots n$

$n-1$

$\vdots$

$\lambda_{n-2} \quad (n-(n-2)+1)$

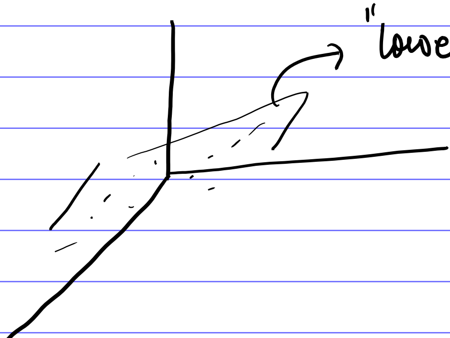
λ_{n-1}
λ_n Eigenspace automatically determined.

Total is $n + n-1 + \cdots + 3 + \underbrace{1}_{\lambda_n, \lambda_{n-1}}$ ___

remaining two parameters.

I haven't thought about how to make this properly rigorous. I'm open to ideas.

→ Ex. Show this hermitian matrices with repeated evals have codim 3.



So in some sense, generic eigenvalue will always avoid multiple eigenvalues; i.e., repulsion is built in.

So a "generic" hermitian matrix has distinct eigenvalues.

This is called avoidance of crossing, and you can find more information in Lax's

"Linear Algebra"

I think this appears in a paper of Kac and von Neumann. Not sure.

$$\lambda \to \det(\lambda I - A) = \phi(\lambda, A) = \prod_{i=1}^{n} (\lambda - \lambda_i(A))$$

when A is generic, $\Rightarrow$ $\phi'(\lambda_i)$ $i=1,\dots,n$

$$= \prod_{j\neq i} (\lambda_i - \lambda_j) \neq 0$$

Inverse fn theorem says that for f that is continuously differentiable at a with nonzero derivative, then f is actually invertible in a neighborhood around a.

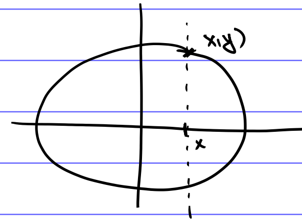
So you can invert ϕ near any of its zeros.

$\Rightarrow$

$A \longmapsto \lambda_i(A)$ are smooth when A has simple spectrum.

Actually you have to use the implicit function theorem.

Take $f(x,y) = x^2+y^2-1$



$\frac{\partial f}{\partial x} = 2x \quad (\neq 0 \text{ iff } x \neq 0)$

$\Rightarrow$ I can express $f(x,y)=0$ as $x = h(y)$ locally. More precisely, locally, $\exists$ a fn $h(y)$ st

$$f(h(y), y) = 0$$

Similarly $\phi(\lambda, A) = \det(\lambda I - A) = 0$

$\frac{\partial \phi}{\partial \lambda} \neq 0$ at any of its zeros. So locally near any of its zeros,

you can write $\lambda = \lambda_i(A)$ (nice functions) λ_i.

Moreover λ_i are differentiable. (see wikipedia)

Similarly you can also make a local differentiable selection $u_i(A)$

Let $A(t)$ be some smooth fn. Think of $\lambda_i(A(t))$ and $u_i(A(t))$ as C' fns of t

$$Au = \lambda u \qquad \dot{A}u + A\dot{u} = \dot{\lambda}u + \lambda\dot{u} \qquad - \star 6$$

subject to $u^* u = 1$

$\Rightarrow \dot{u}^* u + u^* \dot{u} = 0 \Rightarrow u^* \dot{u} = 0$

Multiplying $\star 6$ by u^* we get

$u^* \dot{A} u + u^* A \dot{u} = \dot{\lambda} \Rightarrow u^* \dot{A} u + \cancel{\lambda u^* \dot{u}}^{0} = \dot{\lambda}$

$$\Rightarrow \boxed{\dot{\lambda}_i = u_i^* \dot{A} u_i}$$ reintroduce i

Hadamard 1st variation formula!

If $A(t) = A + tB$ then $\dot{A} = B$

$\Rightarrow \quad \dot{\lambda}_i(t) = u_i^*(t) B u_i(t)$

In fact $|\dot{\lambda}_i(t)| \leq |u_i^* B u_i| \leq \|u_i\| \|Bu_i\| \leq \|B\|_{op}$

This is a Lipschitz map.

We do have to make sure that $A+tB$ has simple spectrum $\forall t$.
If A and B are generic, then so are A and tB?

I dont quite think this is true. Consider

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \qquad B = \begin{bmatrix} -\lambda_1/2 & & \\ & \ddots & \\ & & -\lambda_N/2 \end{bmatrix}$$

For $t = 2$, this thing fails to be generic!

To amend this, let us prove the following:

1) The set M of Hermitian matrices with distinct eigenvalues forms an open and dense set in the space of Hermitian matrices.

2) By previous, the ~~set~~ of matrices with repeated eigenvalues is in a lower dimensional subspace

Ex: The set of matrices with repeated eigenvalues has Lebesgue measure 0! (call it BAD)

Then in any interval $t \in [0,1]$ as long as A and B are generic

$$\lambda_i'(A+tB) = u_i^*(A+tB)\, B\, u_i(A+tB)$$

$$\int_0^1 \lambda_i'(A+tB)\,dt = \int_{(0,1)\setminus BAD} \lambda_i'(A+tB)\,dt$$

But $(0,1)\setminus BAD$ is open $\Rightarrow$ you can write it as a countable disjoint union of open intervals. On each of these intervals

$$|\lambda_i'(A+tB)| \leq \|B\|_{op}$$ is differentiable and has bounded derivative.

By the mean value theorem (f continuous, f' differentiable on (a,b) then $\exists\, c \in (a,b)$ st $f'(c) = \frac{f(b)-f(a)}{b-a}$), f is Lipschitz and hence absolutely continuous.

Ex: We can establish

$$\sum_{i=1}^{n} |\lambda_i(A+B) - \lambda_i(A)|^2 \leq \|B\|_F^2 = \sum_{i,j} |B_{ij}|^2$$

$$\sum_{i=1}^{n} |\lambda_i(A+B) - \lambda_i(A)|^2 \leq \sum_{i=1}^{n} |u_i(A+Bt)^* B\, u_i(A+Bt)|\, \lambda_i(A+Bt)$$

And you can also prove Lidskii this way.

Ex 1.3.13

$$\frac{d^2}{dt^2}\lambda_k = u_k^* \ddot{A} u_k + 2 \sum_{j \neq k} \frac{|u_j^* \dot{A} u_k|^2}{|\lambda_k - \lambda_j|}$$

(locally)

Minors:

$$\begin{bmatrix} \cdot & - - - \\ \vdots & A_{n-1} \end{bmatrix} = A_n$$

Pass from A_n to its principal minor A_{n-1} (another way to perturb matrices)

$$A_{n-1} \text{ has } \lambda_1(A_{n-1}) \geq \lambda_2(A_{n-1}) \geq \cdots \geq \lambda_{n-1}(A_{n-1})$$

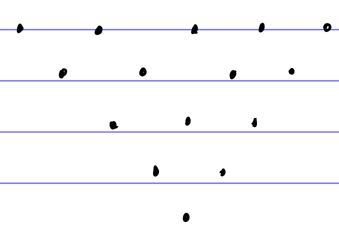
Clearly $\lambda_1(A_{n-1}) \leq \lambda_1(A_n)$

since $$\lambda_1(A_{n-1}) = \sup_{\dim(V)=1} v^* A_{n-1} v$$

Then Cauchy's interlacing property says:

$$\lambda_{i+1}(A_n) \leq \lambda_i(A_{n-1}) \leq \lambda_i(A_n) \qquad 1 \leq i < n$$

In fact if you look at $A_n, A_{n-1}, \ldots$ you can arrange the eigenvalues in a Gelfand–Tsetlin pattern:

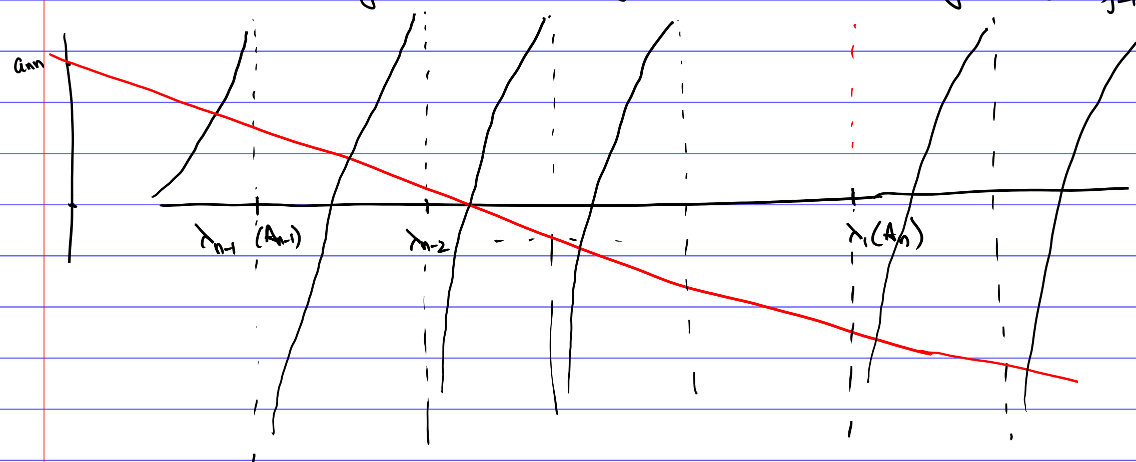


Ex: Eigenvalue equation. Suppose λ is an eigenvalue of A distinct from evals of A_{n-1}.

Then

$$\sum_{j=1}^{n-1} \frac{|u_j(A_{n-1})^* x|^2}{\lambda_j(A_{n-1}) - \lambda} = a_{nn} - \lambda$$

where x is the rightmost column of A minus its last entry $x = (a_{nj})_{j=1}^{n-1}$



How to write what we need?

$$\begin{bmatrix} A_{n-1} & x \\ \bar{x} & a_n \end{bmatrix} \begin{bmatrix} \hat{u} \\ u_n \end{bmatrix} = \lambda \begin{bmatrix} \hat{u} \\ u_n \end{bmatrix}$$

$$A_{n-1}\hat{u} + u_n x = \lambda \hat{u}$$
$$x^*\hat{u} + a_{nn}u_n = \lambda u_n$$

$$u_j^* A_{n-1}\,\hat{u} + (u_j^* X)\,u_n = \lambda\, u_j^* \hat{u}$$

$$\Rightarrow \quad \lambda_j (\hat{u}, u_j) + u_n (X, u_j) = \lambda (\hat{u}, u_j)$$

$$\sum \lambda_j |(\hat{u}, u_j)|^2 + u_n \sum (X, u_j)\overline{(\hat{u}, u_j)} = \lambda (1 - |u_n|^2)$$

using $\sum |(\hat{u}, u_j)|^2 = |\hat{u}|^2 = 1 - |u_n|^2$

$$\lambda_j (\hat{u}, u_j)\overline{(X, u_j)} + u_n |(X, u_j)|^2 = \lambda (\hat{u}, u_j)\overline{(X, u_j)}$$

$$\Rightarrow \cancel{u_n} \sum \frac{|(X, u_j)|^2}{\lambda - \lambda_j} = \sum (\hat{u}, u_j)\overline{(X, u_j)} = (\hat{u}, X)$$

$$= \cancel{u_n} (\lambda - a_n)$$

$$(\hat{u}, X) = \left(\sum (\hat{u}, u_j) u_j, \sum (X, u_j) u_j\right) = \sum \overline{(X, u_j)}\,(\hat{u}, u_j)$$

You see how eigenvalue repulsion is built into the eigenvalue equation?

Singular value decomposition

let $0 \leq p \leq n$ and let A be a linear transformation from an n dimensional complex Hilbert space (U) to a p dim space (V).

$A_{n\times p}$ matrix. Then $\exists$ $\sigma_1(A) \geq \sigma_2(A) \cdots \sigma_p(A) \geq 0$

and $(u_1(A) \cdots, u_p(A)) \in U$ ON $\quad (v_1(A), \cdots, v_p(A)) \in V$ ON

$$Au_i = \sigma_i v_i \qquad A^* v_i = \sigma_i u_i$$

If $u \in [u_1(A) \oplus \cdots \oplus u_p(A)]^\perp$ then $Au = 0$.

Pf: Induction is on p.

$u \to \|Au\|^2$ when $\|u\| = 1$ it has a maximizer u_1

If $\sigma^2 = \|Au_1\|^2$ Then u_1 is a critical point of

$$\|Au\|^2 - \sigma_1^2 \|u\|^2 \qquad \text{for } u \in \mathbb{C}^d$$

$$u^* A^* A u - \sigma_1^2 u^* u$$

$$= ((u^* A^*)_j (Au)_j - \sigma_1^2 (\bar{u}_j u_j))$$

$$= (\bar{u}_k \bar{A}_{jk} A_{jl} u_l - \sigma_1^2 \bar{u}_j u_j) \qquad u_{r_1} + i u_{r_2} = u_r$$

conjugate does not satisfy CR

You cant complex differentiate this easily. So best to diff wrt u_{r_1}

$$\frac{\partial}{\partial u_{r_1}} = \sum_{l \neq r} \bar{A}_{jr} A_{jl} u_l + \sum_{k \neq r} \bar{u}_k \bar{A}_{jk} A_{jr} + 2u_{r_1} \bar{A}_{jr} A_{jr}$$

$$- \sigma_1^2 2u_{r_1}$$

$$\overline{\sum_{k \neq r} \bar{A}_{jr} A_{jk} u_k} = \sum_{k \neq r} A_{jr} \bar{A}_{jk} \bar{u}_k \qquad u_{r_1}^2 + i u_{r_2}^2$$

$$\Rightarrow \quad \operatorname{Re}\left(\sum_k \bar{A}_{jr} A_{jk} u_k - \sigma_1^2 u_r\right) = 0$$

Similar equation for imaginary part.

$$\Rightarrow \quad A^* A u_1 = \sigma_1^2 u_1$$

let $v_1 = \dfrac{A u_1}{\sigma_1} \quad \Rightarrow \quad A^* v_1 = \sigma_1 u_1$

$$A u_1 = \sigma_1 v_1$$

Take $u \in u_1^\perp$ $\qquad (A^* v_1, u) = 0$

$$\begin{array}{c} (A^* v_1, u) \\ \parallel \\ (v_1, Au) \end{array}$$

Or in matrix notation $\quad u^* A^* v_1 = v_1^* A u = 0$

$\Rightarrow \quad A : u_1^\perp \longrightarrow v_1^\perp \quad$ (Invariant subspace)

So we can restrict A to $u_1^\perp$. By induction hypothesis

$\exists \quad \sigma_2 \geq \cdots \sigma_n \geq 0 \quad (u_2, \ldots, u_n)$ and $(v_2, \ldots, v_n)$

satisfying the conclusions of the theorem.

Take $\max_{u \in u_1^\perp} \|Au\|_2^2 \leq \sigma_1^2$ by definition.

In particular

$$\|Au_k\| = \sigma_k \|v_k\| \leq \sigma_1 \quad \forall k = 2, \ldots, n.$$

Orthogonality also follows by construction ∎

Ex: - Singular values are unique.

- " vectors unique upto scalar mult.

Ex: $\underbrace{\begin{pmatrix} 0 & A_{p\times n} \\ A^*_{n\times p} & 0 \end{pmatrix}}_{n+p} \to$ Hermitian. with eigenvalues $\underbrace{\pm\sigma_1, \ldots, \pm\sigma_p}_{2p}, \underbrace{0, \ldots 0}_{n-p}$

Ex: If $A = A^*$ $\{\sigma_1 \ldots \sigma_n\} = \{|\lambda_1|, \ldots, |\lambda_n|\}$

Also true if A is normal $AA^* = A^*A$. ($n\times n$ matrices)

Ex: $(AA^*)_{p\times p}$ is Hermitian is has eigenvalues

$\sigma_1^2 \ldots \sigma_n^2$

$(A^*A)_{n\times n}$ is Hermitian is has eigenvalues

$\sigma_1^2 \ldots \sigma_n^2, \underbrace{0, \ldots 0}_{n-p \text{ copies.}}$

Ex: Rank of $p\times n$ is the # of its nonzero eigenvalues.

Lots of inequalities are also true:

→ Ky Fan, Weyl, Lidskii, p-Schatten norms.

can deduce all of the Hermitian stuff from this.